O’ZBEKISTON RESPUBLIKASI OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

QARSHI MUHANDISLIK-IQTISODIYOT INSTITUTI

“Oliy matematika” kafedrasi

Mavzu: **Kompleks sonlar va ular ustida amallar**

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1. Mavzu: Kompleks sonlar va ularning geometrik tasviri hamda trigonometrik shakli

1.1. Asosiy ta‘riflar.

Haqiqiy sonlar bilan ish ko’rilganda noldan farqli har qanday haqiqiy sonni kvadrati musbat bo’ladi deyilgan edi. Ammo kvadrati manfiy bo’lgan sonlar bilan ham ish ko’rishga to’g’ri keladi. Bunday sonlar tabiiyki haqiqiy son bo’lmaydi.

Fikrimizning isboti sifatida quyidagi misolni qaraymiz. $x^2+4x+13=0$ kvadrat tenglamani yechimi umumiy formulaga $x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}$ ko‘ra, diskriminant $D=b^2-4ac,\ D=4^2-4\cdot1\cdot13=-26$ manfiy songa teng bo‘lganligi uchun, (haqiqiy sonlar to‘plamida manfiy sondan kvadrat ildiz hisoblab bo‘lmaydi) oddiygina qilib, ildizi mavjud emas deyiladi. Aslida esa haqiqiy sonlar to‘plamida ildizga emas. Bu tenglamani ildizini $x_{1,2}=-2\pm\sqrt{-9}=-2\pm3\cdot\sqrt{-1}$ ko‘rinishida yozib, kvadrati -1 ga teng bo’ladigan son tushunchasini kiritsak, yoki -1 ni kvadrat ildizini biron bir son orqali belgilasak, yuqoridagi tenglama ildizini yozish imkoniyati paydo bo’ladi.

1-ta‘rif. Kvadrati –1 ga teng ifodani **mavhum birlik** deb ataladi va u i orqali belgilanadi. Shunday qilib, $i^2=-1$ yoki $i=\sqrt{-1}$.

Mavhum birlikning ta‘rifidan $i^3=i^2\cdot i=-1\cdot i=i$, $i^4=i^2\cdot i^2=(-1)(-1)=1$, $i^5=i$ va hokazo umuman k butun son uchun $i^{4к}=1$, $i^{4к+1}=i$, $i^{4к+2}=-1$, $i^{4к+3}=-i$ ekannligi kelib chiqadi.

2-ta‘rif. *z* kompleks son deb $z=a+bi$ ko’rinishdagi ifodaga aytiladi, bunda *a* va *b* haqiqiy sonlar. *a* va *b* ni *z* kompleks sonning mos ravishda haqiqiy va mavhum qismlari deyiladi va $Rez=a,\ Jmz=b$ kabi belgilanadi.

Xususiy holda, agar a=0 bo’lsa u holda z=0+ib=bi bo’lib u **sof mavhum** son deyiladi. Agar b=0 bo’lsa z=a+i0=a haqiqiy son hosil bo’ladi. Demak, haqiqiy va sof mavhum sonlar kompleks sonning xususiy holi.

Kompleks son tushunchasidan foydalanib $x^2+4x+13=0$ tenglamaning ildizini $x_{1,2}=-2\pm3i$ ko’rinishda yozamiz.

3-ta‘rif. Ikkita $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlar $a_1=a_2$ $b_1=b_2$ bo’lgandagina teng ($z_1=z_2$) deyiladi.

Demak haqiqiy qismlari o’zaro va mavhum qismlari o’zaro teng bo’lgan kompleks sonlar teng bo’lar ekan.

4-ta‘rif. Ham haqiqiy qismi ham mavhum qismi noldan iborat kompleks son nolga teng deyiladi. Demak, $a=0$, $b=0$ bo’lgandagina $z=0$ va aksincha $z=a+ib=0$ dan $a=0$, $b=0$ kelib chiqadi.

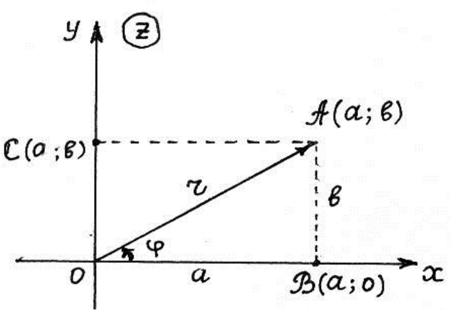
5-ta‘rif. Faqat mavhum qismining ishorasi bilan farq qiluvchi ikkita $z=a+ib$ va $\bar{z}=a-ib$ kompleks sonlar o’zaro **qo’shma** kompleks sonlar deyiladi.

6-ta‘rif. Haqiqiy va mavhum qismlarining ishoralari bilan farq qiluvchi ikkita $z_1=a+ib$ va $z_2=-a-ib=-z_1$ kompleks sonlar **qarama–qarshi** kompleks sonlar deyiladi.

1.2. Kompleks sonning geometrik tasviri

Har qanday $z=a+ib$ kompleks sonni $0xy$ tekislikda koordinatalari a va b bo’lgan $A(a,b)$ nuqta shaklida tasvirlash mumkin. Aksincha, $0xy$ tekislikdagi har qanday $A(a,b)$ nuqtaga $z=a+ib$ kompleks son mos keladi.

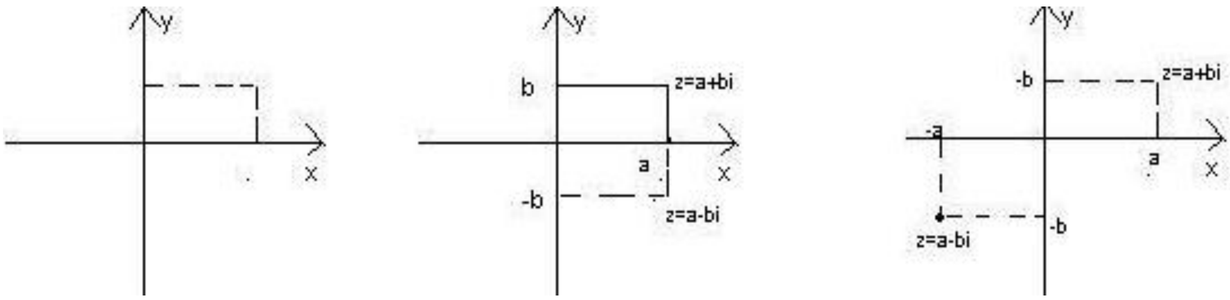
Kompleks sonlar tasvirlanadigan tekislik z kompleks o’zgaruvchining tekisligi deyiladi va tekislikka doiracha ichiga z qo’yiladi. (134-chizma) Shunday qilib kompleks sonning geometrik tasviri Z tekislikning nuqtasidan iborat ekan. $0x$ o’q **haqiqiy o’q**, $0y$ o’q **mavhum o’q** deb ataladi.



1-chizma.

Shunday qilib, kompleks sonning geometrik tasviri Z tekislikdagi nuqtadan yoki vektordan iborat ekan.

Kompleks sonni geometrik shaklidan va yuqoridagi ta’riflardan foydalanib quyidagi fikrlarni keltirishimiz mumkin. Sof mavhum sonlar, $z=0+ib$ mavhum o’qda, haqiqiy sonlar $z=a$ haqiqiy o’qda belgilanadi. O’zaro qo’shma $z=a+ib$ va $z=a-ib$ kompleks sonlar, haqiqiy sonlar o’qiga nisbatan simmetrik joylashgan bo’ladi. Qarama qarshi kompleks sonlar koordinata boshiga nisbatan simmetrik joylashgan bo’ladi.



1.3. Kompleks sonning trigonometrik shakli.

Koordinatalar boshini qutb, 0x o'qning musbat yo'nalishini qutb o'qi deb z kompleks tekislikda qutb koordinatalar sistemasini kiritamiz. φ va r $A(a,b)$ nuqtaning qutb koordinatalari bo'lsin. A nuqtaning qutb radiusi r, ya'ni A nuqtadan qutbgacha bo'lgan masofa $z=a+bi$ kompleks sonning **moduli** deyiladi va $|z|$ kabi belgilanadi. A nuqtaning qutb burchagi φ ni z kompleks sonning **argumenti** deyiladi va Argz kabi belgilanadi. Dekart va qutb koordinatalari orasidagi bog'lanish $a=r\cos\varphi$, $b=r\sin\varphi$ ni hisobga olib $z=a+bi=r\cos\varphi+ir\sin\varphi$ yoki $z=r(\cos\varphi+i\sin\varphi)$ (1) tenglikka ega bo'lamiz.

Bu tenglikning o'ng tomonidagi ifoda $z=a+bi$ kompleks sonning **trigonometrik shakldagi yozuvi** deb ataladi.

Qutb burchagi $\varphi=arctg\frac{b}{a}$ kabi topilishi ma'lum. $\varphi=arctg\frac{b}{a}$ argumentni hisoblashda z kompleks sonning koordinatalar tekisligining qaysi choragida yotishini hisobga olish kerak, chunki $arctg\frac{b}{a}$ qiymatga φ argumentning ikkita qiymatlari mos keladi. Shuning uchun

$$\varphi=\arg z=\begin{cases} arctg\dfrac{b}{a}, & agar \quad a>0, \quad b>0 \quad bo'lsa, \\ \pi+arctg\dfrac{b}{a}, & agar \quad a<0, \quad b \quad istalgan \quad son \quad bo'lsa, \\ 2\pi+arctg\dfrac{b}{a}, & agar \quad a>0, \quad b<0 \quad bo'lsa. \end{cases}$$

tenglikdan foydalanish kerak. Masalan,

arg(1+i)= arctg1=$\frac{\pi}{4}$, chunki a=1>0, b=1>0, arg(-1+i)=π+arctg(-1)=$\pi-\frac{\pi}{4}=\frac{3\pi}{4}$, chunki a=-1<0, b=1>0,

arg(-1-i)= π+arctg1=$\frac{5\pi}{4}$, chunki a=-1<0, b=-1<0,

arg(1-i)= 2π+ arctg(-1)=$2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$, chunki a=1>0, b=-1<0.

Kompleks sonning *z=a+bi* ko'rinishdagi yozuvi kompleks sonning **algebraik** shakli deyiladi.

Kompleks son vektor shaklida tasvirlanganda haqiqiy songa 0*x* o'qda yotuvchi vektor, sof mavhum songa 0*y* o'qda yotuvchi vektor mos keladi.

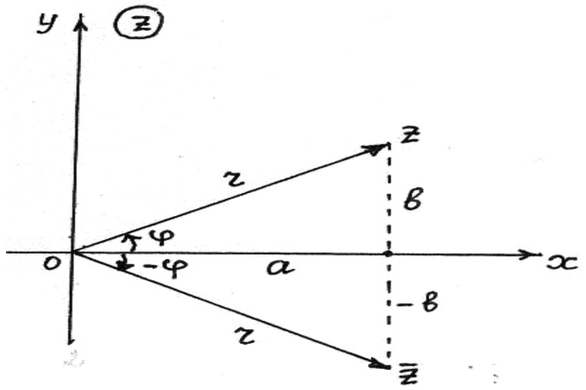
1-misol. $z=a+bi$ va $\bar{z}=a-ib$ qo'shma kompleks sonlar bir xil modullarga ega va argumentlarining absolyut qiymatlari teng, ishoralari qarama-qarshi ekanligini ko'rsating.

Yechish. 2-chizmadan |z|=r=$\sqrt{a^2+b^2}$ va $|\bar{z}|$=r=$\sqrt{a^2+b^2}$ ekani, ya'ni |z|=$|\bar{z}|$ va argz= -arg$\bar{z}$ ekani kelib chiqadi.

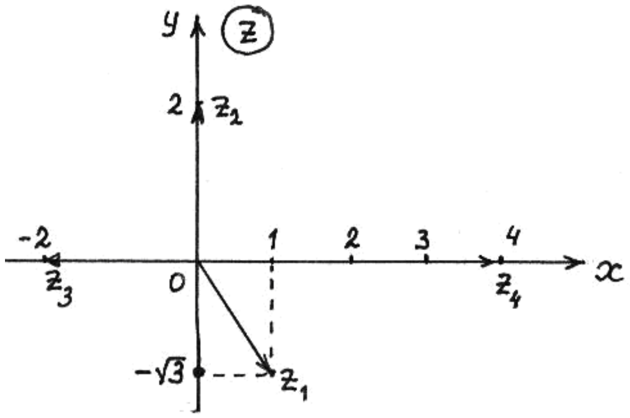
Izoh: Har qanday haqiqiy *A* sonni ham trigonometrik shaklda yozish mumkin, ya'ni *A*>0 bo'lsa, A=A(coso+isino), (2)

A<0 bo'lsa, A=|A|(cosπ+isinπ)

tengliklar o'rinlidir.



2-chizma.



3-chizma.

2-misol. $z_1=1-\sqrt{3}\cdot i$, z_2=2i, z_3=-2, z_4=4 kompleks sonlar trigonometrik shaklda yozilsin.

Yechish. 1) $z_1=1-\sqrt{3}\cdot i$ son uchun a=1, b=$-\sqrt{3}$, $r=\sqrt{1^2+\sqrt{3}^2}=2$,

$$tg\varphi=-\frac{\sqrt{3}}{1}=-\sqrt{3},\quad \varphi=2\pi-arctg\sqrt{3}=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}.$$

Shunday qilib, $z_1=1-\sqrt{3}\cdot i=2$ (cos $\frac{5\pi}{3}$+isin $\frac{5\pi}{3}$).

2) z_2=2i-sof mavhum son. a=0, b=2, r=$\sqrt{0^2+2^2}$ =2, $\varphi=\frac{\pi}{2}$, z_2=2i=2(cos$\frac{\pi}{2}$+isin$\frac{\pi}{2}$).

3) z=-2-manfiy haqiqiy son. Shuning uchun (2) formulaning ikkinchi tenglamasiga binoan z_3=-2=|-2|(cosπ +isinπ) bo’ladi.

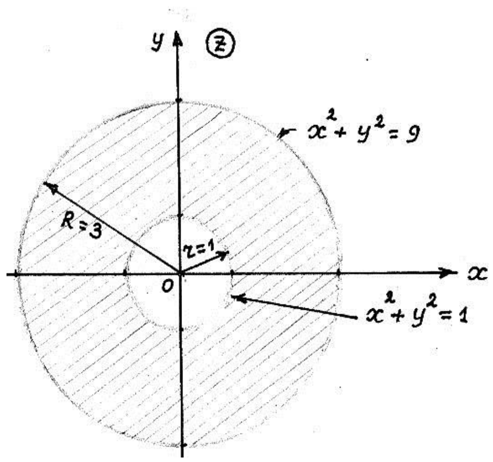
4) z_4=4-musbat haqiqiy son bo’lgani uchun (2) formulaning birinchi tenglamasiga binoan z_4=4=4(coso+isino) bo’ladi.

3-misol. |z|≤3 tengsizlikni qanoatlantiruvchi kompleks sonlarga mos Z- kompleks tekisligi nuqtalarining to’plami topilsin.

Yechish. z=x+iy desak |z|=$\sqrt{x^2+y^2}$ bo’lib, berilgan tengsizlik $\sqrt{x^2+y^2}$ ≤3 yoki $x^2+y^2\le 9$ ko’rinishga ega bo’ladi. $x^2+y^2=9$ tenglik markazi koordinatalar boshida bo’lib radiusi 3 ga teng aylanani ifodalaydi. Demak, $x^2+y^2\le 9$-markazi koordinatalar boshida bo’lib, radiusi 3 ga teng doiraning ichki nuqtalari. Bunda $x^2+y^2=9$ aylananing nuqtalari ham to’plamga tegishli.

4-misol. 1≤|z|<3 tengsizlikni qanoatlantiruvchi z kompleks sonlariga mos Z- kompleks tekisligi nuqtalarining to’plami topilsin.

Yechish. 3-misolning natijasidan foydalanib $1\le x^2+y^2<9$ tengsizliklarga ega bo’lamiz. $x^2+y^2\ge 1$ tengsizlik Z tekislikdagi markazi koordiatalar boshida bo’lib radiusi 1 ga teng aylanada va undan tashqarida yotgan nuqtalar to’plamini ifodalaydi. $x^2+y^2<9$ tengsizlik esa Z tekislikdagi markazi koordinatalar boshida bo’lib radiusi 3 ga teng aylananing ichida yotgan nuqtalar to’plamini ifodalaydi.



(4-chizma).

Demak berilgan tengsizliklar Z tekislikdagi markazi koordinatalar boshida bo’lgan va radiuslari 1 ga va 3 ga teng konsentrik aylanalar orasidagi halqani ifodalar ekan. Bunda radiusi 1 ga teng aylananing nuqtalari ham halqaga tegishli.

5-misol. |z+2-i|=|z+4i| (б) tenglikni qanoatlantiruvchi *z* kompleks sonlar to’plami Z kompleks tekisligida nimani ifodalaydi?

Yechish. z=x+iy desak (б) tenglikni |x+iy+2-i|=|x+iy+4i| yoki

|x+2+i(y-1)|=|x+i(y+4)| ko'rinishda yozish mumkin. Oxirgi tenglikni kompleks sonni modulini topish formulasiga asoslanib $\sqrt{(x+2)^2+(y-1)^2}=\sqrt{x^2+(y+4)^2}$ (в)

kabi yozamiz. Bu yerdagi $\sqrt{(x+2)^2+(y-1)^2}$ ifoda $z=x+iy$ kompleks songa mos keluvchi A(x,y) nuqtadan M(-2;1) nuqtagacha masofani, $\sqrt{(x^2+(y+4)^2}$ esa shu A(x,y) nuqtadan N(0;-4) nuqtagacha masofani ifodalaydi. Demak, (в) tenglik $A(x,y)$ nuqtadan M(-2;1) va N(0;-4) nuqtalargacha masofalar teng ekanligini ko'rsatadi.

Mustaqil yechish uchun mashqlar

1. a) z=3, b) z=2i, d) z=-2, e) z=-3i kompleks sonlar Ⓩ tekisligida vektor ko'rinishida tasvirlansin hamda ularning modullari va argumentlari aniqlansin.

2. Ushbu ifodalar trigonometrik shaklga keltirilsin:

a) 1+i Javob: $\sqrt{2}\left(cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

b) 1-i Javob: $\sqrt{2}\left(cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$.

d) –1+i Javob: $\sqrt{2}\left(cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$.

e) –1-i. Javob: $\sqrt{2}\left(cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$.

f) 3. Javob: 3(cos0+isin0).

g) –4. Javob: 4(cosπ+isinπ).

3. |i-1+2z|≥9 ni qanoatlantiruvchi z kompleks sonlar to'plami Ⓩ kompleks tekisligida nimani ifodalaydi? Javob: Markazi $0_1\left(-\frac{1}{2};\frac{1}{2}\right)$ nuqtada va radiusi R=4,5 bo'lgan aylanada va undan tashqarida yotgan nuqtalar to'plamini ifodalaydi.

4. $z_1=2+3i$ va $z_2=5-4i$ bo'lsa $z=2z_1+3z_2$ hisoblansin.

A) $z=19-6i$ B) $z=19+6i$ C) $z=-19-6i$ D) 13

5. $z=4-3i$ kompleks songa o'zaro qo'shma kompleks sonni toping

A) 16 B) $z=-4-3i$ C) $z=-4+3i$ D) $z=4+3i$

6. $z=-5+2i$ kompleks songa qarama-qarshi sonni ko'psating.

A) $z=-5-2i$ B) $z=5-2i$ C) $\sqrt{29}$ D) 25

7. $z=2+3i$ kompleks songa qarama-qarshi son uchun quyidagilardan qaysi to'g'ri?

A) $0x$ o'qiga nisbatan simmetrik

B) $0y$ o'qiga nisbatan simmetrik

C) koordinatalar boshiga nisbatan simmetrik

D) koordinatalar tekisligiga nisbatan simmetrik

8. $z=3+4i$ songa qo'shma kompleks son uchun quyidagilardan qaysi to'g'ri?

A) $0x$ o'qiga nisbatan simmetrik

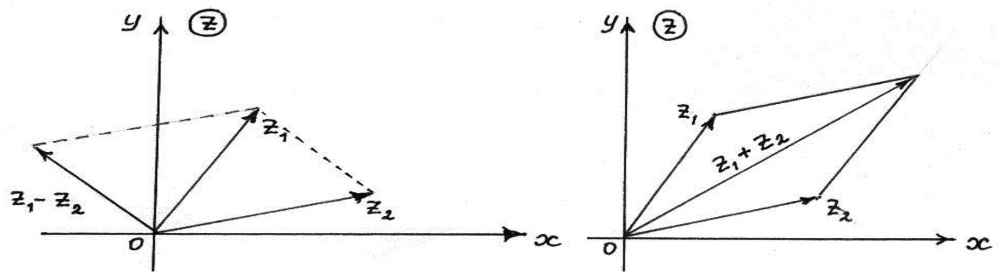
B) $0y$ o'qiga nisbatan simmetrik

C) koordinatalar boshiga nisbatan simmetrik

D) koordinatalar tekisligiga nisbatan simmetrik

2. Mavzu: Kompleks sonlar ustida asosiy amallar.

2.1. Kompleks sonlarni qo'shish. Ikkita $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlarning yig'indisi deb $z_1+z_2=(a_1+ib_1)+(a_2+ib_2)=(a_1+a_2)+i(b_1+b_2)$ (1) tenglik bilan aniqlanuvchi kompleks songa aytiladi. (1) formuladan vektor bilan tasvirlangan kompleks sonlarni qo'shish-vektorlarni qo'shish qoidasiga muvofiq bajarilishi kelib chiqadi. (5^b-chizma)



a) b)

5-chizma.

2.2 Kompleks sonlarni ayirish. Ikkita $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlarning ayirmasi deb shunday kompleks songa aytiladiki, unga z_2 kompleks sonni qo'shganda $z_1$kompleks son hosil bo'ladi (5^a-chizma).

$$z_1-z_2=(a_1+ib_1)-(a_2+ib_2)=(a_1-a_2)+i(b_1-b_2). \quad (2)$$

Ikki kompleks son ayirmasining moduli shu sonlarni z tekisligida tasvirlovchi $A(a_1;b_1)$ va $B(a_2;b_2)$ nuqtalar orasidagi masofaga teng:

$$|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

1-misol. z_1=3+2i va z_2=2-i kompleks sonlarning yig'indisi va ayirmasini toping.

Yechish. z_1+z_2=(3+2i)+(2-i)=(3+2)+ i(2-1)=5+i,

z_1-z_2=(3+2i)-(2-i)=(3-2)+i(2-(-1))=1+3i.

2.3. Kompleks sonlarni ko'paytirish. $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlarning ko'paytmasi deb, $i^2=-1$ ekanligini hisobga olib bu sonlarni ikki had sifatida ko'paytirish qoidasi bo'yicha ko'paytirish natijasida hosil bo'lgan

$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2) \qquad (3)$$

kompleks songa aytiladi.

z_1 va z_2 kompleks sonlar trigonometrik shaklda berilgan bo'lsin: $z_1=r_1(\cos\varphi_1+i\sin\varphi_1)$, $z_2=r_2(\cos\varphi_2+i\sin\varphi_2)$.

Shu kompleks sonlarning ko'paytmasini topamiz.

$z_1\cdot z_2=r_1(\cos\varphi_1+i\sin\varphi_1)\cdot r_2(\cos\varphi_2+i\sin\varphi_2)=r_1\cdot r_2[\cos\varphi_1\cdot\cos\varphi_2+i\cos\varphi_1\sin\varphi_2+i\sin\varphi_1\cdot\cos\varphi_2$ $++i\sin\varphi_1\cdot\sin\varphi_2]=r_1\cdot r_2[(\cos\varphi_1\cdot\cos\varphi_2-\sin\varphi_1\sin\varphi_2)+i(\cos\varphi_1\cdot\sin\varphi_2+\sin\varphi_1\cos\varphi_2)]=$ $=r_1\cdot r_2[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)]$.

Shunday qilib, $z_1\cdot z_2= r_1\cdot r_2[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)]$, (4)

ya'ni ikkita kompleks sonlar ko'paytirnilganda ularning modullari ko'paytiriladi, argumentlari esa qo'shiladi.

2-misol. z_1=3-i va z_2=3-i kompleks sonlarning ko'paytmasi topilsin.

Yechish. $z_1\cdot z_2$=(3-i)(4+2i)=12+6i-4i-2i^2=14+2i.

3-misol. $z_1 = 4\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right)$ va $z_2 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ kompleks sonlarning ko'paytmasi topilsin.

Yechish. (4) formulaga binoan:

$$z_1 z_2 = 4\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right)\cdot 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4\cdot 3\left[\cos\left(\frac{11}{6}\pi + \frac{\pi}{3}\right) + i\sin\left(\frac{11}{6}\pi + \frac{\pi}{3}\right)\right] =$$

$$= 12\left[\cos\left(2\pi + \frac{\pi}{6}\right) + i\sin\left(2\pi + \frac{\pi}{6}\right)\right] = 12\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 12\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 6\sqrt{3} + 6i.$$

4-misol. $z=a+ib$ va $\bar{z} = a - ib$ qo'shma kompleks sonlar ko'paytirilsin.

Yechish. $z\cdot\bar{z}=(a+ib)(a-ib)=a^2-(ib)^2=a^2+b^2$ yoki $z\cdot\bar{z}=|\bar{z}|^2$, chunki $|z|=|\bar{z}|=\sqrt{a^2+b^2}$.

Demak, qo'shma kompleks sonlarni ko'paytmasi haqiqiy son ekan.

2.4 Kompleks sonlarni bo'lish. $z_1=a_1+ib_1$ sonning $z_2=a_2+ib_2$ $(a_2^2+b_2^2\neq 0)$ kompleks soniga bo'linmasi deb z_2 son bilan ko'paytmasi z_1 ga teng $z=x+iy$ kompleks songa aytiladi. Demak $z=\frac{z_1}{z_2}$ va $z_1=z\cdot z_2$ tengliklar teng kuchli.

Kompleks sonni kompleks songa bo'lish amali bo'linuvchi va bo'luvchini bo'luvchining qo'shmasiga ko'paytirish natimjasida amalga oshiriladi:

$$\frac{z_1}{z_2}=\frac{a_1+ib_1}{a_2+ib_2}=\frac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)}=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+i\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}.$$

Agar kompleks sonlar $z_1=r_1(\cos\varphi_1+i\sin\varphi_1)$ va $z_2=r_2(\cos\varphi_2+i\sin\varphi_2)$ trigonometrik shakldа berilgan bo'lsa, u holda:

$$\frac{z_1}{z_2}=\frac{r_1(\cos\varphi_1+i\sin\varphi_1)}{r_2(\cos\varphi_2+i\sin\varphi_2)}=\frac{r_1(\cos\varphi_1+i\sin\varphi_1)\cdot(\cos\varphi_2-i\sin\varphi_2)}{r_2(\cos\varphi_2+i\sin\varphi_2)\cdot(\cos\varphi_2-i\sin\varphi_2)}=$$

$$=\frac{r_1[(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)+i(\sin\varphi_1\cos\varphi_2-\sin\varphi_2\cos\varphi_1)]}{r_2(\cos^2\varphi_2-(i\sin\varphi_2)^2)}=$$

$$=\frac{r_1[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)]}{r_2(\cos^2\varphi_2+\sin^2\varphi_2)}=\frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)].$$

Shunday qilib, $\frac{z_1}{z_2}=\frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)]$, (5)

ya'ni ikkita trigonometrik shakldagi kompleks sonlarni bo'lishda bo'linuvchining moduli bo'luvchining moduliga bo'linadi, bo'linuvchining argumentidan bo'linuvchining argumenti ayriladi.

5-misol. $z_1=3-2i$ kompleks son $z_2=4+i$ songa bo'linsin.

Yechish. $\frac{z_1}{z_2}=\frac{3-2i}{4+i}=\frac{(3-2i)(4-i)}{(4+i)(4-i)}=\frac{3\cdot4-2-i(2\cdot4+3\cdot1)}{4^2+1^2}=\frac{10-i\cdot11}{17}=\frac{10}{17}-\frac{11}{17}i.$

6-misol. $z_1=4\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ kompleks son $z_2=2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ songa bo'linsin.

Yechish. (5) formulaga binoan:

$$\frac{z_1}{z_2}=\frac{4}{2}\left[cos\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)\right]=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2(0+i)=2i.$$

Yuqoridagilardan kelib chiqib quyidagicha xulosa qilishimiz mumkin. Kompleks sonlarni qo'shish va ayirishda ularning algebraic shakldagi, yozuvdagi (1), (2) formulalardan, ko'paytirish va bo'lishda trigonometric shakldagi (4) va (5) formulalardan foydalanish maqsadga muvofiq.

2.5. Kompleks sonni darajaga ko'tarish. Trigonometrik shakldagi kompleks sonlarni ko'paytirish qoidasini ya'ni (4) formulani kompleks sonlar bir nechta umumiy holda n ta bo'lganda ham umumlashtirish mumkin, ya'ni

$$z_1=r_1(\cos\varphi_1+i\sin\varphi_1),$$

$$z_2=r_2(\cos\varphi_2+i\sin\varphi_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$z_n=r_n(\cos\varphi_n+i\sin\varphi_n)$$

sonlarning ko'paytmasi

$z_1 \cdot z_2 \ldots z_n=r_1\cdot r_2\ldots \cdot r_n[\cos(\varphi_1+\varphi_2+\ldots+\varphi_n)+i\sin(\varphi_1+\varphi_2+\ldots+\varphi_n)]$ formula orqali topiladi. Bu formuladan kompleks sonlar o'zaro teng $z_1= z_2=\ldots z_n=z=r(\cos\varphi+ i\sin\varphi)$ bo'lganda

$$z^n=[r(\cos\varphi+i\sin\varphi)]^n=r^n(\cos n\varphi+\sin n\varphi) \qquad (6)$$

formulaga ega bo'lamiz.

Bu formula **Muavr formulasi** deb ataladi. Bu formula kompleks sonni biror natural darajaga ko'tarish uchun uning modulini shu darajaga ko'tarish lozimligini, argumentini esa daraja ko'rsatgichiga ko'paytirish kerakligini ko'rsatadi.

7-misol. $(1+i)^{20}$ ni hisoblang.

Yechish. $|z|= r=\sqrt{1^2+1^2}=\sqrt{2}$ $\varphi=arctg\frac{1}{1}=\frac{\pi}{4}$ bo'lgani uchun

$$1+i= z=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$ bo'lib (6) formulaga binoan

$$z^{20}=(1+i)^{20}==\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{20}=\sqrt{2}^{20}\left(\cos 20\cdot\frac{\pi}{4}+i\sin 20\cdot\frac{\pi}{4}\right)=$$

$$=2^{10}(\cos 5\pi+i\sin 5\pi)=1024(\cos\pi+i\sin\pi)=-1024.$$

Muavr formulasida $r=1$ deb olinsa $(\cos\varphi+i\sin\varphi)^n=\cos n\varphi+i\sin n\varphi$ (7) formula kelib chiqadi. Bu formula $\cos n\varphi$, $\sin n\varphi$ funksiyalarni $\cos\varphi$, $\sin\varphi$

funksiyalarning darajalari orqali ifodalash imkonini beradi.

Masalan, n=2 da $(\cos\varphi+i\sin\varphi)^2=\cos 2\varphi+i\sin 2\varphi$ ga ega bo'lamiz, bundan:

$$\cos^2\varphi+2i\cos\varphi\,\sin\varphi+i^2\sin^2\varphi=\cos 2\varphi+i\sin 2\varphi,$$

$$\cos^2\varphi-\sin^2\varphi\;+2i\sin\varphi\cos\varphi=\cos 2\varphi+i\sin 2\varphi.$$

Ikki kompleks sonlarni tengligi shartidan foydalansak

$\cos 2\varphi=\cos^2\varphi-\sin^2\varphi$, $\sin 2\varphi=2\sin\varphi\cdot\cos\varphi$ ma'lum formulalarga ega bo'lamiz.

Shuningdek n=3 da (7) formula $(\cos\varphi+i\sin\varphi)^3=\cos 3\varphi+i\sin 3\varphi$ ko'rinishga ega bo'lib, bundan:

$$\cos^3\varphi+3\cdot\cos^2\varphi\cdot i\sin\varphi+3\cos\varphi(i\sin\varphi)^2+(i\sin\varphi)^3=\cos 3\varphi+i\sin 3\varphi,$$

$$(\cos^3\varphi-3\cdot\cos\varphi\cdot\sin^2\varphi)+i(3\cos^2\varphi\sin\varphi-\sin^3\varphi)=\cos 3\varphi+i\sin 3\varphi.$$

Ikki kompleks sonlarni tengligi shartiga asoslanib

$$\cos 3\varphi=\cos^3\varphi-3\cos\varphi\cdot\sin^2\varphi,\quad \sin 3\varphi=3\cos^2\varphi\sin\varphi-\sin^3\varphi$$

formulalarni hosil qilamiz.

n=4 ga (7) formula $(\cos\varphi+i\sin\varphi)^4=\cos 4\varphi+i\sin 4\varphi$. ko'rinishga ega bo'ladi. Chap tomonni 4 darajaga ko'taramiz.

$(\cos\varphi+i\sin\varphi)^4=[(\cos\varphi+i\sin\varphi)^2]^2=[\cos 2\varphi+i\sin 2\varphi]=\cos^2 2\varphi-\sin^2 2\varphi+2i\sin 2\varphi\cos 2\varphi=$

$(\cos^2 2\varphi-\sin^2 2\varphi)^2-(2\sin\varphi\cos\varphi)^2+2i*2\sin\varphi\cos\varphi(\cos^2\varphi-\sin^2\varphi)=$

$\cos^4\varphi+\sin^4\varphi-6\sin^2\varphi\cos^2\varphi+4i(\sin\varphi\cos^3\varphi-\sin^3\varphi\cos\varphi)$

Bu formulani (7) formulani n=4 dagi qiymatini o'ng tomoniga tenglab $\cos^4\varphi+\sin^4\varphi-6\sin^2\varphi\cos^2\varphi+4i(\sin\varphi\cos^3\varphi-\sin^3\varphi\cos\varphi)=\cos 4\varphi+i\sin 4\varphi$ ga ega bo'lamiz.

Ikki kompleks sonni tengligi shartiga asoslanib,

$\cos 4\varphi=\cos^4\varphi+\sin^4\varphi-6\sin^2\varphi\cos^2\varphi$

$\sin 4\varphi=4(\sin\varphi\cos^3\varphi-\sin^3\varphi\cos\varphi)$

Formulalarni hosil qilamiz. Xuddi shu tarzda (7) formuladan foydalanib, φ ga karrali burchaklarni sinus va cosinus larni φ ga bog'liq formulalarini keltirib chiqarish mumkin.

2.6 Kompleks sondan ildiz chiqarish. z kompleks sonni n–darajali ildizi $\sqrt[n]{z}$ deb n–darajasi ildiz ostidagi songa teng bo'lgan w kompleks songa aytiladi, ya'ni $w^n=z$ bo'lganda $\sqrt[n]{z}=w$ ($n\in N$).

Agar $z=r(\cos\varphi+i\sin\varphi)$ va $w=\rho(\cos\theta+i\sin\theta)$ bo'lsa

$$\sqrt[n]{z(\cos\varphi+i\sin\varphi)}=\rho(\cos\theta+i\sin\theta)$$

tenglik o'rinlidir. Bundan Muavr formulasiga binoan

$z=r(\cos\varphi+i\sin\varphi)=[\rho(\cos\theta+i\sin\theta)]^n=^n\rho(\cos n\theta+i\sin\theta)$ hosil bo'ladi.

Teng kompleks sonlarni modullari teng, argumentlari esa 2π karrali songa farq qilishini hisobga olsak oxirgi tenglikdan $\rho^n=r$, $\pi=\varphi+2p\kappa$ ga ega bo'lamiz . Bundan ρ va θ ni topamiz: $\rho=\sqrt[n]{r}$, $\theta=\dfrac{\varphi+2\pi k}{n}$, bunda к -istalgan butun son, $\sqrt[n]{r}$ -arifmetik ildiz.

Demak, $$w_k=\sqrt[n]{r(\cos\varphi+i\sin\varphi)}=\sqrt[n]{r}(\cos\frac{\varphi+2\pi k}{n}+i\sin\frac{\varphi+2\pi k}{n}). \qquad (8)$$

k ga 0 dan n-1 gacha qiymatlarini berib, ildizning п ta har xil qiymatlarini topamiz. k ning n-1 dan katta qiymatlarida argumentlar topilgan qiymatlardan 2p ga karrali songa farq qiladi va demak, topilgan ildizlar avvalgilar bilan bir xil bo'ladi. Masalan, k=0 va k=n bo'lgangdagi, k=1 va k=n+1 bo'lgandagi va hokazo ildizlar bir xil bo'ladi. Shunday qilib, kompleks sonning n-darajali ildizi п ta har xil qiymatlarga ega bo'lar ekan. Kompleks sonning ildizini topish formulasi (8) ga k=0,1,2,…, n-1deb, yozib qo'yilishi kerak. Shuningdek noldan farqli haqiqiy sonning n-darajali ildizi ham n ta har xil qiymatlarga ega bo'ladi, chunki haqiqiy son kompleks sonning xususiy holi.

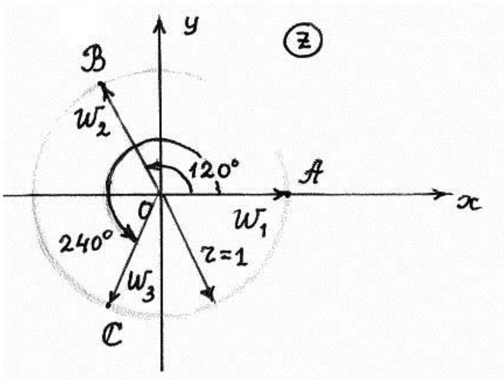
8-misol. $\sqrt[3]{1}$ ning barcha qiymatlari topilsin va ular kompleks tekislikda vektor shaklida tasvirlansin.

Yechish. z=1=1+0i ni trigonometrik shaklda yozamiz. a=1, b=0 bo'lgani uchun $|z|=r=\sqrt{1^2+0^2}=1$, $\varphi=arctg\dfrac{0}{1}=0$ va $z=\cos 0+i\sin 0$ ga ega bo'lamiz.

U holda (8) formula $\sqrt[3]{1}=\sqrt[3]{cos 0+i\sin 0}=\cos\dfrac{2\pi k}{3}+i\sin\dfrac{2\pi k}{3}$ ko'rinishga ega bo'ladi, bunda k=0,1,2. k=0 da $w_1=\cos 0+i\sin 0=1$, k=1 da $w_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$,

k=2 da $w_3=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=\cos\left(\pi+\frac{\pi}{3}\right)+i\sin\left(\pi+\frac{\pi}{3}\right)=-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

w_1, w_2 va w_3 kompleks sonlarning barchasini moduli 1 ga teng ekanligini hisobga olib markazi koordinatalar boshida bo’lib radiusi 1 ga teng aylana yasaymiz. Boshi koordinatalar boshida bo’lib uchi shu aylanada yotgan, hamda 0x o’qning musbat yo’nalishi bilan 0^0,120^0 va 240^0 $\left(0,\frac{2\pi}{3}va\frac{4\pi}{3}\right)$ burchak tashkil etuvchi $\overrightarrow{OA}$, $\overrightarrow{OB}$ va $\overrightarrow{OC}$ vektorlar mos ravishda w_1, w_2 va w_3 kompleks sonlarining geometrik tasviri bo’ladi. (6-chizma).



6-chizma.

Shunday qilib, $\sqrt[3]{1}$ ning uchta qiymati $\sqrt[3]{1}$ =1+io; $\sqrt[3]{1}$ =-$\left(\frac{1}{2}\right)$+ i$\frac{\sqrt{3}}{2}$; $\sqrt[3]{1}$ = -$\left(\frac{1}{2}\right)$- i$\frac{\sqrt{3}}{2}$.

2.7. Ikki hadli tenglamalarni yechish. z^n=A ko’rinishdagi tenglama **ikki hadli tenglama** deyiladi, bunda *A* aniq kompleks son. Shu tenglamaning ildizlarini topamiz.

a) *A* kompleks son bo’lsin. Bu holda (8) formulaga binoan tenglamaning ildizlari

$$z_k=\sqrt[n]{A}=\sqrt[n]{|A|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right) \quad (9)$$

formula yordamida topiladi, bunda φ=argA, k=0,1,2,…n-1.

b) A musbat haqiqiy son bo’lsin. U holda φ=argA=0 bo’lib (9) formula

$$z_k=\sqrt[n]{A}\cdot\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right) \quad (10)$$

ko’rinishini oladi (k=0,1,2,,...,n-1)

d) A manfiy haqiqiy son bo’lsin. U holda φ=argA=p bo’lganligi sababli (9) formuladan $z_k=\sqrt[n]{|A|}\left(\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n}\right)$ (11)

hosil bo’ladi. Xususiy holda z^n=1 tenglamaning barcha ildizlari

$$z_k=\sqrt[n]{1}=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n} \quad (12)$$

formula yordamida, $z^n=-1$ tenglamaning barcha ildizlari

$$z_k=\sqrt[n]{-1}=\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n} \quad (13)$$

formula yordamida topiladi (k=0,1,2, n-1).

9-misol. $z^4=1$ tenglama yechilsin.

Yechish. (12) formulaga binoan $z_k=\cos\frac{2k\pi}{4}+i\sin\frac{2k\pi}{4}==\cos\frac{k\pi}{2}+i\sin\frac{k\pi}{2}$

bo'ladi. k o'rniga 0,1,2,3 qiymatlarni qo'yib ushbularni topamiz:

$z_0=\cos 0+i\sin 0=1$,

$z_1=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i=i$,

$z_2=\cos p+i\sin p=-1$,

$z_3=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i$. Javob: $z_0=1$, $z_1=i$, $z_2=-1$, $z_3=-1$.

Mustaqil yechish uchun mashqlar.

1. (3+2i)+(2-i) topilsin. Javob: 5+i.
2. (4+3i)-(6-4i) topilsin. Javob: -2+7i.
3. (3+2i)(2-3i) topilsin. Javob: 12-5i.
4. (3+5i)(4-i) topilsin. Javob: 17+17i.
5. $\frac{3-i}{4+5i}$ topilsin. Javob: $\frac{7}{41}-\frac{19}{41}i$.
6. $\frac{1-i}{-2-2i}$ topilsin. Javob: $\frac{1}{2}i$.
7. $(4-7i)^3$ topilsin. Javob: -524+7i.
8. $(2(\cos 18^o+i\sin 18^o))^5$ topilsin. Javob: 32i.
9. Quyidagi algebraik shakldagi kompleks sonlarni trigonometrik shaklga keltirib, so'ngra Muavr formulasini qo'llang.

a) $(1+i)^{10}$; b) $(1-i)^{16}$; d) $(\sqrt{3}+i)^{20}$; e) $(\sqrt{3}-i)^{30}$; f) $(1+\cos б+i\sin б)^{п}$.

10. $z_1=3(\cos 20^o-i\sin 20^o)$ son $z_2=2(\cos 10^o-i\sin 10^o)$ songa bo'linsin. Javob: $\frac{3}{4}\left(\sqrt{3}-i\right)$.

11. $\sqrt[3]{-8}$ topilsin. Javob: $1+i\sqrt{3}$; -2; $1-i\sqrt{3}$.

12. $z_k = \sqrt[3]{-\sqrt{2}+i\sqrt{3}}$ topilsin. Javob: $z_1 = \sqrt[3]{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$,

$z_2 = \sqrt[3]{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$, $z_3 = \sqrt[3]{2}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$.

13. $z^3 = i$ tenglama yechilsin. Javob: $z_1 = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$; $z_2 = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$; $z_3 = 1$.

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